ON THE THEORY OF STABILITY OF PROCESSES WITH DISTRIBUTED PARAMETERS

(K TEORII USTOICHIVOSTI PROTSESSOV S RASPREDELENNYMI PARAMETRAMI)

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Authors of [1 and 2] have shown that in the investigation of the stability of processes with distributed parameters, Liapunov functions should be replaced with functionals. A method of constructing these functionals is, however, not given. In the present paper we show, that the functionals analogous to Liapunov functions can be constructed for the investigation of the stability of solution of a system of linear integro-differential equations and, that they appear as integral quadratic forms. The problem of constructing these functionals is thus reduced to solving a boundary value problem.

For nonlinear systems of integro-differential equations, stability theorems in the first approximation are proved.

An analogous problem for a system of partial differential equations and plane fluid flow, is solved in [3 and 4].

1. Let us consider the process described by the following system of partial integro-differential equations

$$\frac{\partial \varphi_i}{\partial t} = L_i(\varphi) + \int_{\tau_z} \sum_{p=1}^{n} b_{ip}(x, z)\, \varphi_p(z, t)\, d\tau_z \qquad (i = 1, 2, \ldots, n_1) \tag{1.1}$$

$$\int_{\tau_z} K_i(x, z) \frac{\partial \varphi_i}{\partial t}\, d\tau_z = L_i(\varphi) + \int_{\tau_z} \sum_{p=1}^{n} b_{ip}(x, z)\, \varphi_p(z, t)\, d\tau_z \qquad (i = n_1 + 1, \ldots, n)$$

$$L_i(\varphi) \equiv \sum_{j=1}^{n} \left[a_{ij}(x)\, \varphi_j(x, t) + \sum_{p=1}^{m} a_{ij}{}^{p}(x) \frac{\partial \varphi_j}{\partial x_p} + \sum_{p,q=1}^{m} a_{ij}{}^{pq}(x) \frac{\partial^2 \varphi_j}{\partial x_p\, \partial x_q} \right] \tag{1.2}$$

$$x \equiv (x_1, x_2, \ldots, x_m), \qquad z \equiv (z_1, z_2, \ldots, z_m), \qquad \varphi \equiv (\varphi_1, \varphi_2, \ldots, \varphi_n) \tag{1.3}$$

Here t is the time, φ is the vector function describing state of the process and $x_1, x_2, \ldots, x_m$ are the coordinates in the region τ within which the process is taking place. When the integration with respect to z is performed, then the region τ is denoted by τ_z.

The coefficients $a_{ij} = a_{ij}(x)$, $a_{ij}{}^{p} = a_{ij}{}^{p}(x)$, $a_{ij}^{pq} = a_{ij}^{pq}(x)$ are continuous and twice differentiable with respect to the coordinates $x_1, x_2, \ldots, x_m$. The kernels $K_i = K_i(x, z)$ and $b_{ip} = b_{ip}(x, z)$ are regular and $K_i = K_i(x, z)$ are assumed to be symmetrical and closed.

We also assume that the system (1.1) contains derivatives with respect to x of up to the second order inclusive. The methods of construction are applicable to other cases.

The functions $\varphi_i(x, t)$, $(i = 1, 2, \ldots, n)$ satisfy the homogeneous boundary conditions, e.g.

$$\sum_{j=1}^{n}\left[A_{ij}(x)\varphi_j(x,t) + \sum_{p=1}^{n} A_{ij}^{p}(x)\frac{\partial\varphi_j(x,t)}{\partial x_p}\right] = 0 \qquad (x \in S) \qquad (1.4)$$

where S is the surface bounding the region τ.

Some of the coefficients A_{ij} and A_{ij}^{p} may be equal to zero and this depends on the problem, e.g. when $A_{ij}^{p} \equiv 0$, then the boundary conditions reduce to $(\varphi_j)_S = 0$ $(j = 1, \ldots, n)$.

We assume that for the given initial $\varphi_0 = \varphi(x, t = t_0)$ and boundary conditions (1.4), the system (1.1) has a unique solution for $t \geq t_0$. When the process is unperturbed, we have $\varphi \equiv 0$, $t \geqslant t_0$, $x \in \tau$. Initial conditions differ for perturbed and unperturbed motion. In the present case the initial conditions are kept unperturbed. We should note that, when we speak of perturbing the initial conditions, we understand that the perturbations act continuously. Such a case is dealt with in the investigation of the stability of plane fluid flow in [4].

We shall introduce $\rho = \rho[\varphi]$ representing some positive functional, as the measure of stability. It will characterize the behavior of the system in the mean over the region τ at any instant of time $t \geq t_0$. In a number of cases, additional constraints imposed on the initial conditions by means of another measure $\rho_0 = \rho_0[\varphi]$, are found advantageous. At the same time we assume that $\rho \leq c\rho_0$, where c is a positive constant.

The process $\varphi \equiv 0$ is called stable in both, ρ and ρ_0, if, for any positive ϵ such $\delta = \delta(\epsilon) > 0$ can be found that $\rho \leq \epsilon$, $(t \geq t_0)$ when $\rho_0 < \delta(\epsilon)$, $(t = t_0)$.

A nonperturbed process $\varphi \equiv 0$ is called asymptotically stable in both measures if it is stable in ρ and ρ_0 and if $\rho \to 0$ as $t \to \infty$.

When the stability in one measure is considered, we assume that $\rho_0 \equiv \rho$.

2. Functionals representing the analogues of Liapunov functions, satisfy the following functional equation

$$dv / dt = u \qquad (2.1)$$

Here the derivative with respect to t is calculated according to (1.1) with (1.4) taken into account, while u and v are functionals.

We shall first construct the solution of (2.1), as a linear integral form

$$v = \int_{\tau_x}\left\{\sum_{i=1}^{n_1} f_i(x)\varphi_i(x,t) + \sum_{i=n_1+1}^{n} f_i(x)\int_{\tau_z} K_i(x,z)\varphi_i(z,t)\,d\tau_z\right\}d\tau_x \qquad (2.2)$$

Its derivative with respect to t will be

$$\frac{dv}{dt} = \int_{\tau_x}\left\{\sum_{i=1}^{n_1} f_i(x)\frac{\partial\varphi_i(x,t)}{\partial t} + \sum_{i=n_1+1}^{n} f_i(x)\int_{\tau_z} K_i(x,z)\frac{\partial\varphi_i(z,t)}{\partial t}\,d\tau_z\right\}d\tau_x$$

Utilizing (1.1) and integrating by parts, we obtain

$$\frac{dv}{dt} = \int_{\tau_x} \sum_{j=1}^{n} \varphi_j(x,t) \left\{ L_{xj}^*(f_1, f_2, \ldots, f_n) + \int_{\tau_z} \sum_{i=1}^{n} b_{ij}(z,x) f_i(z)\, d\tau_z \right\} d\tau_x + $$
$$+ \int_{S_x} Q\, dS_x \qquad (2.3)$$

$$L_{xj}^*(f_1, f_2, \ldots, f_n) \equiv \sum_{i=1}^{n} \left[a_{ij}(x) f_i - \sum_{p=1}^{m} \frac{\partial a_{ij}^{p}(x) f_i}{\partial x_p} + \sum_{p,q=1}^{m} \frac{\partial^2 a_{ij}^{pq}(x) f_i}{\partial x_p \partial x_q} \right]$$

$$Q \equiv \left\{ \sum_{i,j=1}^{n} \left[\sum_{p=1}^{m} \varphi_j \left(a_{ij}^{p} f_i - \sum_{q=1}^{m} \frac{\partial a_{ij}^{pq} f_i}{\partial x_q} \right) \cos(n, x_p) + \right. \right.$$
$$\left. \left. + \sum_{p,q=1}^{m} a_{ij}^{pq} \frac{\partial \varphi_j}{\partial x_p} f_i \cos(n, x_q) \right] \right\}_{S_x} \qquad (2.4)$$

where S_x is the surface S when the variable of integration is denoted by x.

Some of the terms in Q will, by virtue of the boundary conditions (1.4), be equal to zero. Let us choose, for the functions $f_1(x)$, such boundary conditions that $Q = 0$. Let

$$u = \int_{\tau_x} \sum_{j=1}^{n} u_i(x) \varphi_i(x,t)\, d\tau_x \qquad (2.5)$$

Insertion of (2.3) and (2.5) into (2.1), yields

$$L_{xj}^*(f_1, f_2, \ldots, f_n) + \int_{\tau_z} \sum_{i=1}^{n} b_{ij}(z,x) f_i(z)\, d\tau_z = u_j(x) \qquad (j = 1, 2, \ldots, n) \qquad (2.6)$$

which represents the system of equations defining the functions $f_1 = f_1(x)$ with $u_1 = u_1(x)$ ($i = 1, 2, \ldots, n$) given.

Now, Equation (2.1) will satisfy the integral quadratic form

$$v = \int_{\tau_x} \int_{\tau_\xi} \sum_{i,j=1}^{n} f_{ij}(x, \xi) \psi_i(x,t) \psi_j(\xi,t)\, d\tau_x\, d\tau_\xi \qquad (2.7)$$

where

$$\psi_i = \varphi_i \quad (i = 1, 2, \ldots, n_1); \qquad \psi_i = \int_{\tau_z} K_i(x,z) \varphi_i(z,t)\, d\tau_z \quad (i = n_1 + 1, \ldots, n)$$

Let us find dv/dt in accordance with (1.1)

$$\frac{dv}{dt} = \int_{\tau_x} \int_{\tau_\xi} \sum_{i,j=1}^{n} f_{ij}(x,\xi) \left[\frac{\partial \psi_i}{\partial t} \psi_j + \psi_i \frac{\partial \psi_j}{\partial t} \right] d\tau_x\, d\tau_\xi =$$
$$= \int_{\tau_x} \int_{\tau_\xi} \sum_{i,j=1}^{n} L^*_{ij} [f_{11}(x,\xi), \ldots, f_{nn}(x,\xi)]\, \varphi_i(x,t) \varphi_j(\xi,t)\, d\tau_x\, d\tau_\xi +$$
$$+ \int_{\tau_x} \sum_{i=1}^{n} \psi_i(x,t) \int_{S_\xi} Q_i'\, dS_\xi\, d\tau_x + \int_{\tau_\xi} \sum_{j=1}^{n} \psi_j(\xi,t) \int_{S_x} Q_j''\, dS_x\, d\tau_\xi \qquad (2.8)$$

where (2.9)

$$L^*_{ij}[f_{11}(x,\xi),\ldots,f_{nn}(x,\xi)] = L^*_{xi}[f_{1j}(x,\xi),\ldots,f_{nj}(x,\xi)] + \qquad (i,j=1,\ldots,n_1)$$
$$+ L^*_{\xi j}[f_{i1}(x,\xi),\ldots,f_{in}(x,\xi)] + \int_{\tau_\zeta}\sum_{p=1}^{n}[f_{ip}(x,\zeta)\,b_{pj}(\zeta,\xi) + f_{pj}(\zeta,\xi)\,b_{pi}(\zeta,x)]\,d\tau_\zeta$$

$$L^*_{ij}[f_{11}(x,\xi),\ldots,f_{nn}(x,\xi)] = L^*_{xi}[f_{1j}(x,\xi),\ldots,f_{nj}(x,\xi)] + A_{ij}(x,\xi) +$$
$$+\int_{\tau_z}\sum_{p=1}^{n} f_{pj}(z,\xi)\,b_{pi}(z,x)\,d\tau_z \qquad (i=n_1+1,\ldots,n;\ j=1,2,\ldots,n)$$

$$L^*_{ij}[f_{11}(x,\xi),\ldots,f_{nn}(x,\xi)] = L^*_{\xi j}[f_{i1}(x,\xi),\ldots,f_{in}(x,\xi)] + B_{ij}(x,\xi) +$$
$$+\int_{\tau_\zeta}\sum_{p=1}^{n} f_{ip}(x,\zeta)\,b(\zeta,\xi)\,d\tau_\zeta \qquad (i=1,2,\ldots,n;\ j=n_1+1,\ldots,n)$$

$$L^*_{ij}[f_{11}(x,\xi),\ldots,f_{nn}(x,\xi)] = A_{ij}(x,\xi) + B_{ij}(x,\xi) \qquad (i,j=n_1+1,\ldots,n)$$

$$A_{ij}(x,\xi) = \int_{\tau_z} K_i(z,x)\left\{L_{\xi j}{}^*[f_{i1}(z,\xi),\ldots,f_{in}(z,\xi)] + \int_{\tau_\zeta}\sum_{p=1}^{n} f_{ip}(z,\zeta)b_{pj}(\zeta,\xi)d\tau_\zeta\right\}d\tau_z$$

$$B_{ij}(x,\xi) = \int_{\tau_\zeta} K_j(\zeta,\xi)\left\{L^*_{xi}[f_{1j}(x,\zeta),\ldots,f_{nj}(x,\zeta)] + \right.$$
$$\left. + \int_{\tau_z}\sum_{p=1}^{n} f_{pj}(z,\zeta)\,b_{pi}(z,x)\,d\tau_z\right\}d\tau_\zeta$$

$$Q'_i = \sum_{j=1}^{n}\varphi_i(\xi,t)\sum_{p=1}^{n}\sum_{k=1}^{m}\left[a_{kj}{}^p(\xi)f_{ik} - \sum_{q=1}^{m}\frac{\partial a_{kj}{}^{pq}(\xi)f_{ik}}{\partial\xi_q}\right]\cos(n,\xi_p) +$$
$$+\sum_{k,j=1}^{n}\sum_{p,q=1}^{m} a_{kj}{}^{pq}(\xi)\frac{\partial\varphi_j}{\partial\xi_p}f_{ik}\cos(n,\xi_q)$$

$$Q_j'' = \sum_{i=1}^{n}\varphi_i(x,t)\sum_{p=1}^{m}\sum_{k=1}^{n}\left[a_{ki}{}^p(x)f_{kj} - \sum_{q=1}^{m}\frac{\partial a_{ki}{}^{pq}(x)f_{kj}}{\partial x_q}\right]\cos(n,x_p) +$$
$$+\sum_{k,i=1}^{n}\sum_{p,q=1}^{m} a_{ki}{}^{pq}\frac{\partial\varphi_i}{\partial x_p}f_{kj}\cos(n,x_q)$$

$$L_{xi}{}^*[f_{1j}(x,\xi),\ldots,f_{nj}(x,\xi)] =$$
$$= \sum_{k=1}^{n}\left[f_{kj}\,a_{ki}(x) - \sum_{p=1}^{m}\frac{\partial f_{kj}a_{ki}{}^p(x)}{\partial x_p} + \sum_{p,q=1}^{m}\frac{\partial^2 a_{ki}^{pq}(x)f_{kj}}{\partial x_p\partial x_q}\right]$$

$$L_{\xi j}{}^*[f_{i1}(x,\xi),\ldots,f_{in}(x,\xi)] =$$
$$= \sum_{k=1}^{n}\left[f_{ik}a_{kj}(\xi) - \sum_{p=1}^{m}\frac{\partial f_{ik}\,a_{kj}^{p}(\xi)}{\partial\xi_p} + \sum_{p,q=1}^{m}\frac{\partial^2 a_{kj}{}^{pq}(\xi)f_{ik}}{\partial\xi_p\partial\xi_q}\right]$$

We shall introduce, for the functions $f_{ij} = f_{ij}(x,\xi)$, such boundary conditions that the surface integrals in (2.8) become zero, i.e.

$$(Q_i^1)_{S_\xi} = 0, \qquad (Q_j'')_{S_x} = 0 \tag{2.10}$$

For example, if $(\varphi_i)_{S_x} = 0$, then

$$(f_{ij})_{S_x} = (f_{ij})_{S_\xi} = 0$$

If some integral quadratic form v (2.7) is given, then its derivative will, by virtue of (1.1), be represented in the form (2.8). With the boundary conditions of the functions $f_{ij}(x, \xi)$ satisfying the relations (2.10), the derivative dv/dt will again be an integral quadratic form in $\varphi_i(x, t)$.

Let the functional u be given as an integral quadratic form

$$u = \int_{\tau_x}\int_{\tau_\xi} \sum_{i,j=1}^{n} u_{ij}(x, \xi)\, \varphi_i(x, t)\, \varphi_j(\xi, t)\, d\tau_x d\tau_\xi \tag{2.11}$$

Equation (2.1) is satisfied, when

$$L_{ij}^{*}[f_{11}(x, \xi), \ldots, f_{nn}(x, \xi)] = u_{ij}(x, \xi) \qquad (i, j = 1, 2, \ldots, n) \tag{2.12}$$

are fulfilled. In order to construct the integral quadratic form v (2.7) with u(2.11) given, we must solve the system (2.12) for $f_{ij} = f_{ij}(x, \xi)$ with boundary conditions following from (2.10). In this case, the problem of constructing the functional satisfying (2.1), reduces to the solution of a boundary value problem.

Equation (2.1) can be satisfied by integral forms of higher orders, but the corresponding equations shall not be quoted here.

Note. In some cases, Equation (2.1) can be satisfied by a form, linear or quadratic in $\partial\varphi_i/\partial x_q$, e.g. by the linear form

$$v_1 = \int_{\tau_x} \left\{ \sum_{i=1}^{n_1} \sum_{p=1}^{m} l_i^p(x) \frac{\partial \varphi_i}{\partial x_p} + \sum_{i=n_1+1}^{n} \sum_{p=1}^{m} l_i^p(x) \int_{\tau_z} K_i(x, z) \frac{\partial \varphi^i}{\partial z_p} d\tau_z \right\} d\tau_x$$

or by the form, which is a linear combination of v given by (2.2) and v_1 [3].

Let u (2.11) be a form, positive definite in ρ. We shall denote the class of functions $\{u_{ij}(x, \xi)\}$, $x \in \tau_x$ and $\xi \in \tau_\xi$, satisfying this condition, by K_ρ. If the system (2.12) has a solution with given $u_{ij}(x, \xi)$ belonging to the class K_ρ, then a form v which satisfies (2.1) with u given, always exists.

In the following we shall assume, when considering the sign definite form u, that (2.12) has the solution $f_{ij}(x, \xi)$ and, that the corresponding form v is continuous over ρ.

Below we shall use two theorems, which represent the modifications of theorems first proved in [1 and 2].

Theorem 2.1. If a functional $v = v[\varphi]$ continuous and sign definite over ρ exists for (1.1) and if its derivative with respect to time is, by virtue of (1.1), also sign definite, but of the sign opposite to that of $v = v[\varphi]$, then the process $\varphi \equiv 0$ is asymptotically stable in ρ.

Theorem 2.2. If a functional $v = v[\varphi]$ continuous over the measure ρ exists for (1.1) and if its derivative with respect to time is, by virtue of (1.1), sign definite while the functional itself is not of constant sign opposite to that of $u = u[\varphi]$, then the solution $\varphi \equiv 0$ is unstable in ρ.

When the stability of processes with distributed parameters is investigated, then, by the Theorems 2.1 and 2.2, the sign definiteness and continuity of the functionals $\mathcal{V}$ or $\mathcal{U}$ (analogues of Liapunov functions) must be checked. For one particular case, the criterion of sign definiteness is given in the Appendix. If the function $f_{ij}(x, \xi)$ with an integrable square is the solution of (2.12), then the form $\mathcal{V}$ (2.7) will be continuous over the measure

$$\rho = \int_{\tau_x} \sum_{i=1}^{n} \varphi_i^2 \, d\tau_x$$

which is easily confirmed by applying the Cauchy-Buniakowski inequality.

The measure of stability ρ is found a priori from the physical considerations. In isolated cases, adoption of the functional $\mathcal{V}$ (2.7) as the measure ρ is of interest. We find that in this case the check on the sign definiteness and continuity ceases to be necessary and the magnitude $\mathcal{V} > 0$ characterizes the behavior of the process in the mean over the region T , with respect to time .

3 . Let us consider a nonlinear system of integro-differential equations

$$\frac{\partial \varphi_i}{\partial t} = L_i(\varphi) + \int_{\tau_z} \sum_{p=1}^{n} b_{ip}(x, z)\, \varphi_p(z, t)\, d\tau_z + \Phi_i \qquad (i = 1, 2, \ldots, n_1)$$

$$\int_{\tau_z} K_i(x, z) \frac{\partial \varphi_i}{\partial t} d\tau_z = L_i(\varphi) + \int_{\tau_z} \sum_{p=1}^{n} b_{ip}(x, z)\, \varphi_p(z, t)\, d\tau_z + \Phi_i$$

$$(i = n_1 + 1, \ldots, n) \qquad (3.1)$$

Here $\Phi_i = \Phi_i(x, t, \varphi, \ldots)$ are functions nonlinear in $\varphi \equiv \varphi(\varphi_1, \varphi_2, \ldots, \varphi_n)$ and in its derivatives with respect to coordinates $x_1, x_2, \ldots, x_m$, and first terms of their expansions are of the order higher than the first. We shall call the system (1.1), the first approximation equations.

Theorem 3.1 . Solution $\varphi \equiv 0$ of the nonlinear system (3.1) is stable in ρ, if it as asymptotically stable in the first approximation, if the functional $\mathcal{V}$ (2.7) is continuous in ρ and if the condition

$$u + \Delta u \leqslant 0, \quad \text{for} \quad \left|\frac{\Delta u}{u}\right| \leqslant \varepsilon < 1 \qquad (3.2)$$

is fulfilled.

Here (3.3)

$$\Delta u = \int_{\tau_x}\int_{\tau_\xi} \sum_{i,j=1}^{n} f_{ij}(x, \xi)\, [\varphi_i(x, t)\, \Phi_j(\xi, t, \varphi, \ldots) + \Phi_i(x, t, \varphi, \ldots)\, \varphi_j(\xi, t)]\, d\tau_x d\tau_\xi$$

and $\mathcal{U}$ is a negative definite form defined by (2.11).

Proof . Consider the integral quadratic form $\mathcal{V}$ satisfying the equation $\mathcal{V}^{\bullet} = \mathcal{U}$, where the right-hand side is a negative definite integral form (2.11) and the derivative $\mathcal{V}^{\bullet}$ is computed according to the linear system of the first approximation (1.1) . By definition of the theorem, $\mathcal{V}$ is continuous in ρ and the solution $\varphi \equiv 0$ of (1.1) is asymptotically stable. Moreover, three following variants are possible : form $\mathcal{V}$ may assume negative values, form $\mathcal{V}$ may be permanently positive or, it may be positive definite. The functional $\mathcal{V}$ cannot assume negative values, since in that case Theorem

2.2 would be valid and the process would be unstable. Assume that $v \geq 0$ when $\rho \neq 0$. Let us consider the process with initial state $\rho \neq 0$, for which $v = 0$. But $v^{\bullet} < 0$. Hence, in the course of the process we should obtain $v < 0$ which is impossible. Since by definition v is nonnegative, it can only be positive definite.

Let us now construct the derivative of this form with respect to time and according to the nonlinear equations (3.1) ; the result will be $v^{\bullet} = u + \Delta u \leq 0$. Form v is continuous and positive definite, while $u + \Delta u$ is nonpositive. Consequently, the solution $\varphi \equiv 0$ of the nonlinear system (3.1) is stable. If $u + \Delta u$ is negative definite, then the process $\varphi \equiv 0$ is asymptotically stable.

Theorem 3.2. Solution $\varphi \equiv 0$ of the nonlinear system (3.1) will be unstable in ρ, if it is unstable in the first approximation, if the functional v (2.7) is continuous in ρ and if $u + \Delta u$ (where Δu is given by (3.3)) is a sign definite functional of the same sign as (2.11).

Proof. Let the form v (2.7) satisfy Equation $\dot{v} = u$, where u is a positive definite form of the type of (2.11) and $\dot{v}$ is calculated according to the linear system of the first approximation (1.1). Moreover, the form v cannot be negative definite, since in that case the process $\varphi \equiv 0$ would, according to Theorem 3.1, become asymptotically stable, which contradicts the condition of the Theorem. It also cannot be permanently negative, since that would give $v = 0$ for some $\rho \neq 0$. Assumption that $v^{\bullet} > 0$ implies that $v > 0$ and that contradicts the initial assumption that $v \leq 0$. Thus, the integral form v will not be of constant sign opposite to that of u.

Let us now compute the derivative $v^{\bullet}$ according to the complete system (3.1) and represent it in the form $\dot{v} = u + \Delta u$. Here $u + \Delta v$ is a sign definite, e.g. a positive definite functional, while v is continuous and is not of constant, and opposite to that of $u + \Delta u$, sign. Consequently, the solution $\varphi \equiv 0$ of (3.1) will be unstable.

4. Examples. (1). Let us consider a process defined by

$$\frac{\partial \varphi}{\partial t} = \int_0^l b(x, \xi)\, \varphi(\xi, t)\, d\xi \qquad \left(b(x, \xi) = \sum_{i=1}^{\infty} b_i \psi_i(x)\, \psi_i(\xi) \right) \tag{4.1}$$

Here $\psi_i = \psi_i(x)$ is a complete, orthonormalized system of functions in the interval $[0, l]$. We shall represent the solution of (4.1), as

$$\varphi = \sum_{i=1}^{\infty} a_i(t)\, \psi_i(x)$$

and we shall investigate the stability over the measure

$$\rho \equiv v \equiv \int_0^l \varphi^2(x, t)\, dx = \sum_{i=1}^{\infty} a_i^2(t) \tag{4.2}$$

Let us find the derivative of v in accordance with (4.1)

$$\frac{dv}{dt} = 2 \int_0^l \int_0^l b(x, \xi)\, \varphi(x, t)\, \varphi(\xi, t)\, d\xi\, dx = 2 \sum_{i=1}^{\infty} b_i a_i^2$$

If $b_i \leq 0$, then the process $\varphi = 0$ is stable. If, on the other hand, $b_i < 0$ and $\lim b_i \leq 0$ as $t \to \infty$, then $v^{\bullet}$ is negative definite (see Appendix). Moreover, the process $\varphi \equiv 0$ becomes asymptotically stable.

2) . Consider a nonlinear equation describing the vibrations of a string in a resisting medium
$$\frac{\partial^2\varphi}{\partial t^2}=a\frac{\partial^2\varphi}{\partial x^2}+b\frac{\partial\varphi}{\partial t}+\Phi\left(x,t,\varphi,\frac{\partial\varphi}{\partial t}\right),\qquad \varphi(0,t)=\varphi(\pi,t)=0 \tag{4.3}$$
where $\varphi=\varphi(x,t)$ is the deflection of the string from its equilibrium position $\varphi\equiv 0$. Introducing the notation
$$\varphi_1=\frac{\partial\varphi}{\partial t},\qquad \varphi_2=\varphi$$
we obtain (4. 3) in reduced form
$$\frac{\partial\varphi_1}{\partial t}=a\frac{\partial^2\varphi_2}{\partial x^2}+b\varphi_1+\Phi(x,t,\varphi_2,\varphi_1),\qquad \frac{\partial\varphi_2}{\partial t}=\varphi_1 \tag{4.4}$$
Let us consider the stability over the measure
$$\rho=\int_0^\pi\left[\left(\frac{\partial\varphi}{\partial t}\right)^2+\left(\frac{\partial\varphi}{\partial x}\right)^2\right]dx=\int_0^\pi\left[\varphi_1^2+\left(\frac{\partial\varphi_2}{\partial x}\right)^2\right]dx \tag{4.5}$$
and the integral form
$$u=\int_0^\pi\int_0^\pi\left[u_{11}(x,\xi)\varphi_1(x,t)\varphi_1(\xi,t)+u_{22}(x,\xi)\frac{\partial\varphi_2(x,t)}{\partial x}\frac{\partial\varphi_2(\xi,t)}{\partial\xi}\right]dx\,d\xi \tag{4.6}$$
With $\varphi_2(0,t)=\varphi_2(\pi,t)=0$, we obtain
$$u=\int_0^\pi\int_0^\pi\left[u_{11}(x,\xi)\varphi_1(x,t)\varphi_1(\xi,t)+\frac{\partial^2u_{22}(x,\xi)}{\partial x\,\partial\xi}\varphi_2(x,t)\varphi_2(\xi,t)\right]dx\,d\xi$$
where u_{11} and u_{22} are given by
$$u_{11}=\sum_{s,\,r=1}^{\infty}u_{sr}'\sin sx\sin r\xi,\qquad u_{22}=\sum_{s,\,r=1}^{\infty}u_{sr}''\cos sx\cos r\xi \tag{4.7}$$
We shall consider the integral form v represented by
$$v=\int_0^\pi\int_0^\pi\left\{f_{11}\varphi_1(x,t)\varphi_1(\xi,t)+f_{12}\varphi_1(x,t)\frac{\partial\varphi_2(\xi,t)}{\partial\xi}+\right.$$
$$\left.+f_{21}\frac{\partial\varphi_2(x,t)}{\partial x}\varphi_1(\xi,t)+f_{22}\frac{\partial\varphi_2(x,t)}{\partial x}\frac{\partial\varphi_2(\xi,t)}{\partial\xi}\right\}dx\,d\xi \tag{4.8}$$
Let
$$F_{11}=f_{11},\qquad F_{12}=-\frac{\partial f_{12}}{\partial\xi},\qquad F_{21}=-\frac{\partial f_{21}}{\partial x},\qquad F_{22}=\frac{\partial^2 f_{22}}{\partial x\partial\xi} \tag{4.9}$$
The function $\varphi_2(x,t)$ becomes zero at the ends of the interval $[0,\pi]$, hence the boundary conditions f_{ij} are arbitrary .

Putting $\varphi_2(0,t)=\varphi_2(\pi,t)=0$, we obtain
$$v=\int_0^\pi\int_0^\pi\{F_{11}\varphi_1(x,t)\varphi_1(\xi,t)+F_{12}\varphi_1(x,t)\varphi_2(\xi,t)+$$
$$+F_{21}\varphi_2(x,t)\varphi_1(\xi,t)+F_{22}\varphi_2(x,t)\varphi_2(\xi,t)\}\,dx\,d\xi$$
Integral form v exists if the system (2. 12) which, in this case, can be written as
$$F_{21}+F_{12}+2bF_{11}=u_{11},\qquad bF_{12}+F_{22}+\frac{\partial^2F_{11}}{\partial\xi^2}=0$$
$$bF_{21}+F_{22}+\frac{\partial^2F_{11}}{\partial x^2}=0,\qquad \frac{\partial^2F_{12}}{\partial x^2}+\frac{\partial^2F_{21}}{\partial\xi^2}=\frac{\partial^2u_{22}}{\partial x\,\partial\xi}$$
has a solution.

By (2. 10), we have the following boundary conditions :

$$F_{11}(x,\xi)|_{\substack{x=\pi,0\\ \xi=\pi,0}} = F_{12}(x,\xi)|_{x=\pi,0} = F_{21}(x,\xi)|_{\xi=\pi,0} = 0 \qquad (4.10)$$

This system can be represented as

$$\frac{\partial^2 F_{11}}{\partial \xi^2} + \frac{\partial^2 F_{11}}{\partial x^2} + 2F_{22} - 2b^2 F_{11} = -bu_{11}, \qquad F_{12} = -\frac{1}{b}\left(F_{22} + \frac{\partial^2 F_{11}}{\partial \xi^2}\right) \qquad (4.11)$$

$$2\frac{\partial^4 F_{11}}{\partial \xi^2 \partial x^2} + \frac{\partial^2 F_{22}}{\partial x^2} + \frac{\partial^2 F_{22}}{\partial \xi^2} = -b\frac{\partial^2 u_{22}}{\partial x \partial \xi}, \qquad F_{21} = -\frac{1}{b}\left(F_{22} + \frac{\partial^2 F_{11}}{\partial x^2}\right) \qquad (4.12)$$

and we shall seek the solution of the above equations in the form

$$F_{11} = \sum_{s,r=1}^{\infty} A_{sr} \sin sx \sin r\xi, \qquad F_{22} = \sum_{s,r=1}^{\infty} B_{sr} \sin sx \sin r\xi \qquad (4.13)$$

Let us insert (4.13) into (4.11) and (4.12). Taking (4.7) into account, we obtain the following system of algebraic equations for A_{sr} and B_{sr}

$$-(s^2 + r^2 + 2b^2) A_{sr} + 2B_{sr} = -bu_{sr}'$$

$$2s^2r^2 A_{sr} - (s^2 + r^2) B_{sr} = -bsru_{sr}''$$

which yield

$$A_{sr} = \frac{-b(s^2+r^2)u_{sr}' - 2bsru_{sr}''}{4s^2r^2 - (s^2+r^2)(s^2+r^2+2b^2)}$$

$$B_{sr} = -\frac{b}{2}u_{sr}' - \frac{(s^2+r^2+2b^2)[b(s^2+r^2)u_{sr}' + 2b_{sr}u_{sr}'']}{2[4s^2r^2 - (s^2+r^2)(s^2+r^2+2b^2)]} \qquad (4.14)$$

$$(s, r = 1, 2, \ldots)$$

Expression (4.13) defines the functions F_{11} and F_{22}, while (4.11) and (4.12) define F_{12} and F_{21}

$$F_{12} = -\frac{1}{b}\sum_{s,r=1}^{\infty} (B_{sr} - r^2 A_{sr}) \sin sx \sin r\xi$$

$$F_{21} = -\frac{1}{b}\sum_{s,r=1}^{\infty} (B_{sr} - s^2 A_{sr}) \sin sx \sin r\xi$$

Having determined F_{11}, F_{12}, F_{21} and F_{22} we can, by means of (4.8) and (4.9), determine the functional v. Let

$$u_{sr}' = u_s'\delta_{sr}, \quad u_{sr}'' = u_s''\delta_{sr}.$$

Then $A_{sr} = B_{sr} = 0$ when $s \neq r$, and

$$A_{ss} = \frac{1}{2b}(u_s' + u_s''), \qquad B_{ss} = \frac{1}{2b}[s^2 u_s' + (s^2 + b^2)u_s'']$$

Let us determine

$$f_{11} = \sum_{s=1}^{\infty} A_{ss} \sin sx \sin s\xi, \qquad f_{12} = -\frac{1}{b}\sum_{s=1}^{\infty}\left(\frac{B_{ss}}{s} - sA_{ss}\right)\sin sx \cos s\xi$$

$$f_{21} = -\frac{1}{b}\sum_{s=1}^{\infty}\left(\frac{B_{ss}}{s} - sA_{ss}\right)\cos sx \sin s\xi, \qquad f_{22} = \sum_{s=1}^{\infty}\frac{B_{ss}}{s^2}\cos sx \cos s\xi$$

next. This is equivalent to

$$f_{11} = \frac{1}{2b}\sum_{s=1}^{\infty}(u_s' + u_s'')\sin sx \sin s\xi, \qquad f_{12} = -\frac{1}{2b}\sum_{s=1}^{\infty}\frac{u_s''}{s}\sin sx \cos s\xi$$

$$f_{21} = -\frac{1}{2b}\sum_{s=1}^{\infty}\frac{u_s''}{s}\cos sx \sin s\xi, \qquad f_{22} = \frac{1}{2b}\sum_{s=1}^{\infty}\left[u_s' + \left(1 + \frac{b^2}{s^2}\right)u_s''\right]\cos sx \cos s\xi$$

We shall represent the solution of equation of vibration of a string by

$$\varphi_1 = \sum_{s=1}^{\infty} \alpha_s \sin sx, \qquad \varphi_2 = \sum_{s=1}^{\infty} \beta_s \sin sx \tag{4.15}$$

In case of linearized equations we have

$$\alpha_s = a_s \cos st + b_s \sin st, \qquad \beta_s = -sa_s \sin st + sb_s \cos st$$

where a_s and b_s are constant coefficients. Then

$$\rho = \frac{\pi}{2} \sum_{s=1}^{\infty} (\alpha_s^2 + s^2\beta_s^2) \tag{4.16}$$

The process $\varphi \equiv 0$ is stable in the first approximation over ρ and the functional

$$v = \int_0^{\pi}\int_0^{\pi} \frac{1}{2b} \sum_{s=1}^{\infty} \Big\{(u_s' + u_s'') \sin sx \sin r\xi \varphi_1(x,t)\, \varphi_1(\xi,t) -$$

$$- \frac{u_s''}{s} \sin sx \cos s\xi \varphi_1(x,t) \frac{\partial \varphi_2(\xi,t)}{\partial \xi} - \frac{u_s''}{s} \cos sx \sin s\xi \frac{\partial \varphi_2(x,t)}{\partial x} \varphi_1(\xi,t) +$$

$$+ \left[u_s' + \left(1 + \frac{b^2}{s^2}\right) u_s''\right] \cos sx \cos s\xi \frac{\partial \varphi_2(x,t)}{\partial x} \frac{\partial \varphi_2(\xi,t)}{\partial \xi}\Big\} dx\, d\xi$$

or

$$v = \frac{\pi^2}{8b} \sum_{s=1}^{\infty} \left\{(u_s' + u_s'') \alpha_s^2 - 2u_s'' \alpha_s \beta_s + s^2 \left[u_s' + \left(1 + \frac{b^2}{s^2}\right) u_s''\right] \beta_s^2\right\}$$

If the coefficients u_s' and u_s'' are bounded, then the form v is bounded and continuous over ρ. Indeed, we have the following estimate :

$$|v| \leqslant \frac{1}{2|b|} \max \left| u_r' + \left(2 + \frac{b^2}{r^2}\right) u_r'' \right| \sum_{s=1}^{\infty} (\alpha_s^2 + |\alpha_s \beta_s| + s^2\beta_s^2) \leqslant$$

$$\leqslant \text{const} \left\{ \sum_{s=1}^{\infty} (\alpha_s^2 + s^2\beta_s^2) + \left(\sum_{s=1}^{\infty} \alpha_s^2 \sum_{r=1}^{\infty} \beta_r^2 \right)^{1/2} \right\} \tag{4.17}$$

Taking into account

$$\left(\sum_{s=1}^{\infty} \alpha_s^2 \sum_{r=1}^{\infty} \beta_r^2 \right)^{1/2} \left(\sum_{s=1}^{\infty} (\alpha_s^2 + s^2\beta_s^2) \right)^{-1} \leqslant 1 \tag{4.18}$$

we obtain

$$|v| \leqslant \text{const}\, \rho$$

Let us now consider the negative definite form u. When $u_s'/b > 0$ and $u_s''/b > 0$ we have the inequality

$$v \geqslant v_0 = \frac{1}{8b} \sum_{s=1}^{\infty} \{(u_s' + u_s'') \alpha_s^2 - 2u_s'' \alpha_s \beta_s + s^2 (u_s' + u_s'') \beta_s^2\} \tag{4.19}$$

Consequently, the Sylvester criterion

$$\begin{vmatrix} (u_s' + u_s'')/b & -u_s''/b \\ -u_s''/b & [s^2(u_s' + u_s'')]/b \end{vmatrix} = s^2 \left(\frac{u_s' + u_s''}{b}\right)^2 - \left(\frac{u_s''}{b}\right)^2 > 0$$

is fulfilled for all $s = 1, 2, \ldots$, and v_0 can be expressed as a series with positive terms. If also $\rho \geq \epsilon > 0$, then a number $\delta(\epsilon) > 0$ exists such, that $v \geq v_0 > \delta(\epsilon)$ (see Appendix). Consequently, the form v is positive definite. Now consider

$$u_{11}(x,\xi) = -\delta(x - \xi), \quad u_{22}(x,\xi) = -\delta(x - \xi), \quad \text{i.e.} \quad u_s' = u_s'' = -1$$

Let $b < 0$ and let v be a bounded, continuous and positive definite form over ρ and

$$u = -\int_0^\pi \left[\varphi_1^2 + \left(\frac{\partial \varphi_2}{\partial x}\right)^2\right] dx = -\rho$$

If the nonlinearity ϕ is such that $|\Delta u| / |u| \leqslant \varepsilon < 1$, where

$$\Delta u = \int_0^\pi\int_0^\pi \Big\{ f_{11}\left[\varphi_1(x,t)\,\phi(\xi,t,\ldots) + \phi(x,t,\ldots)\,\varphi_1(\xi,t)\right] +$$
$$+ f_{12}\phi(x,t,\ldots)\frac{\partial\varphi_2(\xi,t)}{\partial\xi} + f_{21}\,\frac{\partial\varphi_2(x,t)}{\partial x}\phi(\xi,t,\ldots)\Big\}\,dx\,d\xi$$

then the process $\varphi \equiv 0$ will be stable on ρ with nonlinear terms taken into consideration. For example, we have the estimate

$$|\Delta u| \leqslant \max\{|f_{11}|, |f_{12}|, |f_{21}|, |f_{22}|\}\left\{\int_0^\pi |\varphi_1| + \right.$$
$$\left. + \int_0^\pi \left|\frac{\partial\varphi_2}{\partial x}\right| dx\right\}\int_0^\pi |\phi|\,dx \leqslant \operatorname{const}\rho^{1/2}\int_0^\pi |\phi|\,dx$$

If

$$|\phi| \leqslant A_1\varphi_1^2 + A_2\left(\frac{\partial\varphi_2}{\partial x}\right)^2$$

where A_1 and A_2 are constants, then $|\Delta u| / |u| \leqslant \operatorname{const}\rho^{1/2}$. With ρ sufficiently small, the functional $u + \Delta u$ will be sign definite and of the sign opposite to that of v. Consequently, the process $\varphi \equiv 0$ will be asymptotically stable with nonlinear terms of (4.3) taken into account.

5. Appendix. Let the measure

$$\rho_0 = \int_\tau \sum_{i=1}^n \varphi_i^2\,d\tau, \qquad t \geqslant t_0 \tag{5.1}$$

be given and functions φ_1 be represented by the series

$$\varphi_i = \sum_{k=1}^\infty \alpha_k^{\,i}(t)\,\psi_k(x), \quad \alpha_k^{\,i} = \alpha_k^{\,i}(t) = \int_\tau \varphi_i(x,t)\,\psi_k(x)\,d\tau, \quad x\in\tau, \quad t \geqslant t_0 \tag{5.2}$$

Here $\psi_k = \psi_k(x)$ is a complete normalized orthogonal system of functions in the region τ. Consider the form u given by

$$u = \sum_{k=1}^\infty\sum_{i=1}^n u_i^{\,k}(\alpha_k^i)^2 \tag{5.3}$$

For the measure ρ_0 we have

$$\rho_0 = \sum_{k=1}^\infty\sum_{i=1}^n (\alpha_k^i)^2 \tag{5.4}$$

We shall only consider such an aggregate $\{\alpha_k^{\,i}\}$ for which ρ_0 is bounded, i.e. a series whose general term $(\alpha_k^{\,i})^2$ converges as $k\to\infty$.

If $u_1^{\,k} > 0$ and $\lim u_1^{\,k} \geq 0$ as $k\to\infty$, then the form u will be positive definite over ρ_0. Indeed, let $\epsilon > 0$ and $\rho_0 \geq \epsilon > 0$ be given.

The series (5.4) is convergent, hence we can say that the final term of (5.4) will be smaller than any positive number ϵ_0, e.g. $\epsilon_0 = \frac{1}{2}\epsilon$, provided that N is sufficiently large

$$\sum_{k=1}^{N-1}\sum_{i=1}^{n}(\alpha_k^i)^2 = \rho_0 - \sum_{k=N}^{\infty}\sum_{i=1}^{n}(\alpha_k^i)^2 \geqslant \varepsilon - \varepsilon_0 = \frac{1}{2}\varepsilon$$

$$\sum_{k=1}^{\infty}\sum_{i=1}^{n}u_i^k(\alpha_k^i)^2 \geqslant \sum_{k=1}^{N-1}\sum_{i=1}^{n}u_i^k(\alpha_k^i)^2 \geqslant$$

$$\geqslant \min(u_i^k)\sum_{k=1}^{N-1}\sum_{i=1}^{n}(\alpha_k^i)^2 \geqslant \min(u_i^k)\frac{\varepsilon}{2} = \delta(\varepsilon) > 0$$

where N is dependent on ϵ. Thus u is positive definite over ρ_0. For any positive number ϵ there exists a number $\delta = \delta(\epsilon) > 0$ such, that $u \geq \delta(\epsilon)$ when $\rho_0 > \epsilon$.

We should also note that the form

$$u = \sum_{k=1}^{\infty}\sum_{i,j=1}^{n}u_{ij}^k\alpha_k^i\alpha_k^j$$

is positive definite over ρ_0, if

$$u_k = \sum_{i,j=1}^{n}u_{ij}^k\alpha_k^i\alpha_k^j$$

is positive definite for any fixed $k \geq 1$ and when $k \to \infty$, or, when $u_{ij}^k \to 0$ as $k \to \infty$.

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